Relation between the Fourier coefficients and the interaction effects expressed by using Kronecker product in the experimental design

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Abstract

In this paper, I present that the relation between the Fourier coefficients and the interaction effects in the experimental design can be more easily expressed by using Kronecker product. From the result, the interaction effects can be easily obtained from the Fourier coefficients. Therefore, it is possible to implement the estimation procedures easily as well as to understand how any interaction affects the response variable in the model based on an orthonormal system.

Key Words:
Kronecker product, orthonormal system, Fourier transform

1 Introduction

In most areas of scientific research, experimentation is a major tool for acquiring new knowledge or a better understanding of the target phenomenon. Experiments usually aim to study how changes in various factors affect the response variable of interest [1]. Since the model used most often at present in experimental design is expressed through the effect of each factor, it is easy to understand how each factor affects the response variable [2, 5, 6]. However, since the model contains redundant parameters and is not expressed in terms of an orthonormal system, it often takes much time to implement the procedure for estimating the effects.

On the other hand, it has recently been shown that the model in experimental design can also be expressed in terms of an orthonormal system [7, 8, 11][1]. In this case, the model is expressed by using Fourier coefficients instead of the effect of each factor. As there is an abundance of software for calculating the Fourier transform, such a system allows for a straightforward implementation of the procedures for estimating the Fourier coefficients by using Fourier transform. Moreover, the effects as for the general mean, the main factor and the interaction can be obtained from the computed Fourier coefficients because the relation has been obtained [9, 10]. However, because the relation about the interaction are complicated, it is often difficult to obtain the interaction effects from the Fourier coefficients.

In this paper, I present that the relation can be more easily expressed by using Kronecker product. From this result, any interaction effects can be easily obtained from the Fourier coefficients. Therefore, it is possible to implement the estimation procedures easily as well as to understand how any interaction affects the response variable in the model based on an orthonormal system.

[1]The model of [7, 8] is different from that of previous works (the model expressed through the effect of each factor), but the applications are the same.

2 Preliminaries

2.1 Fourier Analysis on Finite Abelian Groups

Here I give a brief explanation of Fourier analysis on finite Abelian groups. Characters play an important role in the context of finite Fourier series.

2.1.1 Characters [4]

Let G be a finite Abelian group (with the multiplicative notation) and S^1 the unit circle in the complex plane. A character on G is a complex valued function $\mathcal{X} : G \to S^1$ which satisfies the condition

$$\mathcal{X}(\boldsymbol{x} \cdot \boldsymbol{x}') = \mathcal{X}(\boldsymbol{x})\mathcal{X}(\boldsymbol{x}') \quad \forall \boldsymbol{x}, \boldsymbol{x}' \in G. \tag{1}$$

In other words, a character is a homomorphism from G to the circle group.

2.1.2 Fourier Transform [3]

Let $G_i, i = 1, 2, \ldots, n$, be Abelian groups of orders $|G_i| = g_i, i = 1, 2, \ldots, n, g_1 \leq g_2 \leq \cdots \leq g_n$, and

$$G = \times_{i=1}^{n} G_i \quad and \quad g = \prod_{i=1}^{n} g_i. \tag{2}$$

We write the group operations additively $G_i \equiv \langle G_i, + \rangle$. Because the character group of G is isomorphic to G, we can index the characters by the elements of G, i.e., $\{\mathcal{X}_{\boldsymbol{a}}(\boldsymbol{x}) | \boldsymbol{a} \in G\}$ are the characters of G. Note that $\mathcal{X}_{\boldsymbol{0}}(\boldsymbol{x})$ is the principal character and it is identically equal to 1. The characters $\{\mathcal{X}_{\boldsymbol{a}}(\boldsymbol{x}) | \boldsymbol{a} \in G\}$ form an orthonormal system:

$$\frac{1}{g} \sum_{\boldsymbol{x} \in G} \mathcal{X}_{\boldsymbol{a}}(\boldsymbol{x}) \mathcal{X}^*_{\boldsymbol{b}}(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{a} = \boldsymbol{b}, \\ 0, & \boldsymbol{a} \neq \boldsymbol{b}, \end{cases} \tag{3}$$

where $\mathcal{X}^*_{\boldsymbol{b}}(\boldsymbol{x})$ is the complex-conjugate of $\mathcal{X}_{\boldsymbol{b}}(\boldsymbol{x})$.

Any function $f : G \to \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, can be uniquely expressed as a linear combination of the characters:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in G} f_{\boldsymbol{a}} \mathcal{X}_{\boldsymbol{a}}(\boldsymbol{x}), \tag{4}$$

where the complex number

$$f_{\boldsymbol{a}} = \frac{1}{g} \sum_{\boldsymbol{x} \in G} f(\boldsymbol{x}) \mathcal{X}^*_{\boldsymbol{a}}(\boldsymbol{x}), \tag{5}$$

is the $\boldsymbol{a}$-th *Fourier coefficient* of f.

2.2 Fourier Analysis on $GF(q)^n$ [8]

Suppose q is a prime power. Let $GF(q)$ be a Galois field of order q, where a Galois field is a field that contains finite elements. We also use $GF(q)^n$ to denote the set of all n-tuples with entries from $GF(q)$. The elements of $GF(q)^n$ is referred to as vectors .

Example 1 *Consider* $GF(3) = \{0, 1, 2\}$ *and* $n = 5$. *Then*

$$GF(3)^5 = \{00000, 10000, \cdots, 22222\} \tag{6}$$

and $|GF(3)^5| = 243$. □

Specifying the group G in Sect.2.1.2 to be the support group of $GF(q)^n$ and $g = q^n$, the relations (3), (4) and (5) also hold over $GF(q)^n$ domain.

2.3 Kronecker product

In this subsection, I introduce the definition of Kronecker product.

Definition 1 *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be respectively* $m \times n$ *and* $u \times v$ *matrices. Their* Kronecker product, *denoted by* $A \otimes B$, *is the* $mu \times nv$ *matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{bmatrix}, \tag{7}$$

where $a_{ij}B$ *stands for the* $u \times v$ *matrix with entries* $a_{ij}b_{rs}(1 \leq r \leq u, 1 \leq s \leq v)$. □

3 Experimental Design

In this section, I provide a short introduction to experimental design. For a detailed explanation, refer to [6].

3.1 Experimental Design Model [6]

Let $F_1, F_2, \ldots, F_n$ denote the n factors to be included in an experiment. The levels of each factor can be represented by $GF(q)$, and the level combinations can be represented by the n-tuples $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in GF(q)^n$.

Example 2 *Let Machine (F_1) and Worker (F_2) be factors that might influence the quantity of a product. Suppose each factor has two levels.*

F_1 : new machine (level 0), old machine (level 1).

F_2 : skilled worker (level 0), unskilled worker (level 1).

For example, $\boldsymbol{x} = 01$ represents a combination of new machine and unskilled worker.

Then, the effect of the machine, averaged over both workers, is referred to as the effect of main factor F_1. Similarly, the effect of the worker, averaged over both machines, is referred to as the effect of main factor F_2. And the contrast between the effect of the machine for an unskilled worker and the effect of the machine for a skilled worker is referred to as the effect of the interaction of F_1 and F_2. $\square$

Let the set $A \subseteq \{0,1\}^n$ represent all factors that might influence the response of the experiment. The *Hamming weight* $w(\boldsymbol{a})$ of a vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in A$ is defined as the number of nonzero components. The main factors are represented by $MF = \{i | a_i = 1, \boldsymbol{a} \in A_1\}$, where $A_1 = \{\boldsymbol{a} | w(\boldsymbol{a}) = 1, \boldsymbol{a} \in A\}$. The interactive 2-factors are represented by $IF_2 = \{\{i,j\} | a_i = 1, a_j = 1, \boldsymbol{a} \in A_2\}$, where $A_2 = \{\boldsymbol{a} | w(\boldsymbol{a}) = 2, \boldsymbol{a} \in A\}$. the interactive 3-factors are represented by $IF_3 = \{\{i,j,k\} | a_i = 1, a_j = 1, a_k = 1, \boldsymbol{a} \in A_3\}$, where $A_3 = \{\boldsymbol{a} | w(\boldsymbol{a}) = 3, \boldsymbol{a} \in A\}$. The interaction between more than three factors is also represented similarly.

Example 3 *Consider $A = \{000, 100, 010, 001, 110\}$. Then, $A_1 = \{100, 010, 001\}$ and $MF = \{1, 2, 3\}$, $A_2 = \{110\}$ and $IF_2 = \{\{1, 2\}\}$.*

For example, $1 \in MF$ indicates the main factor of F_1 and $\{1,2\} \in IF_2$ indicates the interactive 2-factor of F_1 and F_2. $\square$

It is usually assumed that the set A satisfies the following monotonicity condition [2].

Definition 2 *Monotonicity*

$$\boldsymbol{a} \in A \rightarrow \boldsymbol{b} \in A \qquad \forall \boldsymbol{b}\ (\boldsymbol{b} \sqsubseteq \boldsymbol{a}), \tag{8}$$

where $(b_1, b_2, \ldots, b_n) \sqsubseteq (a_1, a_2, \ldots, a_n)$ means that if $a_i = 0$ then $b_i = 0, i = 1, 2, \ldots, n$. $\square$

Example 4 *Consider $A = \{00000, 10000, 01000, 00100, 00010, 00001, 11000, 10100, 01100, 10010, 11100\}$.*

Because the set A satisfies (8), A is monotonic. $\square$

Let $y(\boldsymbol{x})$ denote the response of the experiment with level combination $\boldsymbol{x}$ and assume the model

$$\begin{aligned} y(\boldsymbol{x}) &= \mu + \sum_{i \in MF} \alpha_i(x_i) + \sum_{\{i,j\} \in IF_2} \beta_{i,j}(x_i, x_j) \\ &\quad + \sum_{\{i,j,k\} \in IF_3} \gamma_{i,j,k}(x_i, x_j, x_k) + \epsilon_{\boldsymbol{x}}, \end{aligned} \tag{9}$$

where μ is the general mean, $\alpha_i(x_i)$ is the effect of the x_i-th level of Factor F_i, $\beta_{i,j}(x_i, x_j)$ is the effect of the interaction of the x_i-th level of Factor F_i and the x_j-th level of Factor F_j, $\gamma_{i,j,k}(x_i, x_j, x_k)$ is the effect of the interaction of the x_i-th level of Factor F_i, the x_j-th level of Factor F_j and the x_k-th level of Factor F_k and $\epsilon_{\boldsymbol{x}}$ is a random error with zero mean and constant variance σ^2.

Since the model is expressed through the effect of each factor, it is easy to understand how each factor affects the response variable. However, because the constraints $\sum_{\varphi=0}^{q-1}\alpha_i(\varphi) = 0$, $\sum_{\varphi=0}^{q-1}\beta_{i,j}(\varphi,\psi) = 0$, $\sum_{\psi=0}^{q-1}\beta_{i,j}(\varphi,\psi) = 0$, $\sum_{\varphi=0}^{q-1}\gamma_{i,j,k}(\varphi,\psi,\xi) = 0$, $\sum_{\psi=0}^{q-1}\gamma_{i,j,k}(\varphi,\psi,\xi) = 0$, $\sum_{\xi=0}^{q-1}\gamma_{i,j,k}(\varphi,\psi,\xi) = 0$, are assumed, the model contains redundant parameters. Therefore, since the model contains redundant parameters and is not expressed in terms of an orthonormal system, it often takes much time to implement the procedure for estimating the effects.

Example 5 *Consider* $GF(3) = \{0,1,2\}$, $n = 5$ *and* $A = \{00000, 10000, 01000, 00100, 00010, 00001, 11000, 10100, 01100, 10010, 11100\}$.

Then, $\mu, \alpha_1(0), \alpha_1(1), \alpha_1(2), \alpha_2(0), \alpha_2(1), \alpha_2(2), \alpha_3(0), \alpha_3(1), \alpha_3(2), \alpha_4(0), \alpha_4(1), \alpha_4(2), \alpha_5(0), \alpha_5(1), \alpha_5(2), \beta_{1,2}(0,0), \beta_{1,2}(0,1), \beta_{1,2}(0,2), \beta_{1,2}(1,0), \beta_{1,2}(1,1), \beta_{1,2}(1,2), \beta_{1,2}(2,0), \beta_{1,2}(2,1), \beta_{1,2}(2,2), \beta_{1,3}(0,0), \beta_{1,3}(0,1), \beta_{1,3}(0,2), \beta_{1,3}(1,0), \beta_{1,3}(1,1), \beta_{1,3}(1,2), \beta_{1,3}(2,0), \beta_{1,3}(2,1), \beta_{1,3}(2,2), \beta_{2,3}(0,0), \beta_{2,3}(0,1), \beta_{2,3}(0,2), \beta_{2,3}(1,0), \beta_{2,3}(1,1), \beta_{2,3}(1,2), \beta_{2,3}(2,0), \beta_{2,3}(2,1), \beta_{2,3}(2,2), \beta_{1,4}(0,0), \beta_{1,4}(0,1), \beta_{1,4}(0,2), \beta_{1,4}(1,0), \beta_{1,4}(1,1), \beta_{1,4}(1,2), \beta_{1,4}(2,0), \beta_{1,4}(2,1), \beta_{1,4}(2,2),$
$\gamma_{1,2,3}(0,0,0), \gamma_{1,2,3}(0,0,1), \gamma_{1,2,3}(0,0,2),$
$\gamma_{1,2,3}(0,1,0), \gamma_{1,2,3}(0,1,1), \gamma_{1,2,3}(0,1,2),$
$\gamma_{1,2,3}(0,2,0), \gamma_{1,2,3}(0,2,1), \gamma_{1,2,3}(0,2,2),$
$\gamma_{1,2,3}(1,0,0), \gamma_{1,2,3}(1,0,1), \gamma_{1,2,3}(1,0,2),$
$\gamma_{1,2,3}(1,1,0), \gamma_{1,2,3}(1,1,1), \gamma_{1,2,3}(1,1,2),$
$\gamma_{1,2,3}(1,2,0), \gamma_{1,2,3}(1,2,1), \gamma_{1,2,3}(1,2,2),$
$\gamma_{1,2,3}(2,0,0), \gamma_{1,2,3}(2,0,1), \gamma_{1,2,3}(2,0,2),$
$\gamma_{1,2,3}(2,1,0), \gamma_{1,2,3}(2,1,1), \gamma_{1,2,3}(2,1,2),$
$\gamma_{1,2,3}(2,2,0), \gamma_{1,2,3}(2,2,1), \gamma_{1,2,3}(2,2,2),$
are parameters. Hence, the number of parameters is 79. *However, by the constraints, the number of the independent parameters is* 35. □

In experimental design, we are given a model of the experiment. That is, we are given a set $A \subseteq \{0,1\}^n$. First, we determine a set of level combinations $\boldsymbol{x} \in X$, $X \subseteq GF(q)^n$. The set X is called a design. Then, we perform a set of experiments according to the design X and estimate the effects from the result, $\{(\boldsymbol{x}, y(\boldsymbol{x})) | \boldsymbol{x} \in X\}$.

An important standard for evaluating experimental designs is the maximum of the variances of the unbiased estimators of effects calculated from the result of the experiments. It is known that, for a given number of experiments, this criterion is minimized in an orthogonal design [5]. Therefore, there has been much research into orthogonal designs [1, 5].

3.2 Orthogonal Designs [5]

Definition 3 *(Orthogonal Designs)*
Define $v(\boldsymbol{a}) = \{i | a_i \neq 0, 1 \leq i \leq n\}$. *For* $A \subseteq \{0,1\}^n$, *let* H_A *be the* $m \times n$ *matrix*

$$H_A = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1n} \\ h_{21} & h_{22} & \ldots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1} & h_{m2} & \ldots & h_{mn} \end{bmatrix}. \qquad (10)$$

The entries in this matrix, $h_{ij} \in GF(q)$ $(1 \leq i \leq k, 1 \leq j \leq n)$, *satisfy the following conditions.*

1. *The set* $\{\boldsymbol{h}_{\cdot j} | j \in v(\boldsymbol{a}' + \boldsymbol{a}'')\}^2$, *where* $\boldsymbol{h}_{\cdot j}$ *is the* j*-th column of* H_A, *is linearly independent over* $GF(q)$ *for any given* $\boldsymbol{a}', \boldsymbol{a}'' \in A$.
2. *The set* $\{\boldsymbol{h}_{i\cdot} | 1 \leq i \leq m\}$, *where* $\boldsymbol{h}_{i\cdot}$ *is the* i*-th row of* H_A, *is linearly independent over* $GF(q)$.

An orthogonal design $C^{\perp}$ *for main and interactive factors* $A \subseteq \{0,1\}^n$ *is defined by*

$$C^{\perp} = \{\boldsymbol{x} | \boldsymbol{x} = \boldsymbol{r} H_A,\ \boldsymbol{r} \in GF(q)^m\}, \qquad (11)$$

[2] For $\boldsymbol{a}_1 = (a_{11}, a_{12}, \ldots, a_{1n}), \boldsymbol{a}_2 = (a_{21}, a_{22}, \ldots, a_{2n}) \in \{0,1\}^n$, the addition of vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is defined by $\boldsymbol{a}_1 + \boldsymbol{a}_2 = (a_{11} \oplus a_{21}, a_{12} \oplus a_{22}, \ldots, a_{1n} \oplus a_{2n})$, where $\oplus$ is the *exclusive or* operation.

and $|C^{\perp}| = q^m$. □

Many algorithms for constructing H_A have been proposed [1, 5]. However, it is still a very difficult problem to construct H_A when the number of factors n is large and many interactions are included in the model. In this letter, the algorithm is not included because the construction of orthogonal designs is not the purpose of this letter.

3.3 Estimation of the Effects in Experimental Designs [6]

First, I make the following definitions.

$$Y = \sum_{\boldsymbol{x} \in C^{\perp}} y(\boldsymbol{x}), \tag{12}$$

where $|C^{\perp}| = q^m$.

$$Y_i(\varphi) = \sum_{\boldsymbol{x} \in C_i^{\perp}(\varphi)} y(\boldsymbol{x}), \tag{13}$$

where $C_i^{\perp}(\varphi) = \{\boldsymbol{x} | x_i = \varphi, \boldsymbol{x} \in C^{\perp}\}$ and $|C_i^{\perp}(\varphi)| = q^{m-1}$.

$$Y_{i,j}(\varphi, \psi) = \sum_{\boldsymbol{x} \in C_{i,j}^{\perp}(\varphi,\psi)} y(\boldsymbol{x}), \tag{14}$$

where $C_{i,j}^{\perp}(\varphi, \psi) = \{\boldsymbol{x} | x_i = \varphi, x_j = \psi, \boldsymbol{x} \in C^{\perp}\}$ and $|C_{i,j}^{\perp}(\varphi, \psi)| = q^{m-2}$.

$$Y_{i,j,k}(\varphi, \psi, \xi) = \sum_{\boldsymbol{x} \in C_{i,j,k}^{\perp}(\varphi,\psi,\xi)} y(\boldsymbol{x}), \tag{15}$$

where $C_{i,j,k}^{\perp}(\varphi, \psi, \xi) = \{\boldsymbol{x} | x_i = \varphi, x_j = \psi, x_k = \xi, \boldsymbol{x} \in C^{\perp}\}$ and $|C_{i,j,k}^{\perp}(\varphi, \psi, \xi)| = q^{m-3}$.

Let $\bar{y} = \frac{1}{q^m} Y$, $\bar{y}_i(\varphi) = \frac{1}{q^{m-1}} Y_i(\varphi)$, $\bar{y}_{i,j}(\varphi, \psi) = \frac{1}{q^{m-2}} Y_{i,j}(\varphi, \psi)$,
$\bar{y}_{i,j,k}(\varphi, \psi, \xi) = \frac{1}{q^{m-3}} Y_{i,j,k}(\varphi, \psi, \xi)$.

Then, the unbiased estimates of the parameters in (9) are given by

$$\begin{aligned}
\hat{\mu} &= \bar{y}, && (16)\\
\hat{\alpha}_i(\varphi) &= \bar{y}_i(\varphi) - \hat{\mu}, && (17)\\
\hat{\beta}_{i,j}(\varphi, \psi) &= \bar{y}_{i,j}(\varphi, \psi) - \hat{\alpha}_i(\varphi) - \hat{\alpha}_j(\psi) \\
&\quad -\hat{\mu}, && (18)\\
\hat{\gamma}_{i,j,k}(\varphi, \psi, \xi) &= \bar{y}_{i,j,k}(\varphi, \psi, \xi) - \hat{\beta}_{i,j}(\varphi, \psi) \\
&\quad -\hat{\beta}_{i,k}(\varphi, \xi) - \hat{\beta}_{j,k}(\psi, \xi) - \hat{\alpha}_i(\varphi) \\
&\quad -\hat{\alpha}_j(\psi) - \hat{\alpha}_k(\xi) - \hat{\mu}. && (19)
\end{aligned}$$

4 Estimation of the Effects in the Experimental Design using Fourier Transforms

4.1 Experimental Design Model based on an Orthonormal System

We use $y(\boldsymbol{x})$ to denote the response of the experiment with level combination $\boldsymbol{x}$ and assume the model

$$y(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in I_A} f_{\boldsymbol{a}} \mathcal{X}_{\boldsymbol{a}}(\boldsymbol{x}) + \epsilon_{\boldsymbol{x}}, \tag{20}$$

where $I_A = \{(b_1 a_1, \ldots, b_n a_n) | \boldsymbol{a} \in A, b_i \in GF(q)\}$ and $\epsilon_{\boldsymbol{x}}$ is a random error with zero mean and constant variance.

Then, the effects are represented by unknown parameters $\{f_{\boldsymbol{a}} | \boldsymbol{a} \in I_A\}$.

Example 6 *Consider* $GF(3) = \{0, 1, 2\}$, $n = 5$ *and* $A = \{00000, 10000, 01000, 00100, 00010, 00001, 11000, 10100, 01100, 10010, 11100\}$.

Then, $f_{00000}, f_{10000}, f_{20000}, f_{01000}, f_{02000}, f_{00100}, f_{00200}, f_{00010}, f_{00020}, f_{00001}, f_{00002}, f_{11000}, f_{12000}, f_{21000}, f_{22000}, f_{10100}, f_{10200}, f_{20100}, f_{20200}, f_{01100}, f_{01200}, f_{02100}, f_{02200}, f_{10010}, f_{10020}, f_{20010}, f_{20020}, f_{11100}, f_{11200}, f_{12100}, f_{12200}, f_{21100}, f_{21200}, f_{22100}, f_{22200}$, *are parameters. The number of parameters is* 35, *and these parameters are independent.* □

4.2 Estimation of the Effects in Experimental Designs

First, I recall the following theorem.

Theorem 1 *(Sampling Theorem for Bandlimited Functions over a* $GF(q)^n$ *Domain [7])*

Suppose $A \subseteq \{0,1\}^n$ *is monotonic and*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in I_A} f_{\boldsymbol{a}} \mathcal{X}_{\boldsymbol{a}}(\boldsymbol{x}), \tag{21}$$

where $I_A = \{(b_1 a_1, \ldots, b_n a_n) | \boldsymbol{a} \in A, b_i \in GF(q)\}$. *Then, the Fourier coefficients can be computed by*

$$f_{\boldsymbol{a}} = \frac{1}{q^m} \sum_{\boldsymbol{x} \in C^{\perp}} f(\boldsymbol{x}) \mathcal{X}^*_{\boldsymbol{a}}(\boldsymbol{x}), \tag{22}$$

where $C^{\perp}$ *is an orthogonal design for* A *(*$|C^{\perp}| = q^m$*).* □

When we experiment according to the orthogonal design $C^{\perp}$, we can obtain unbiased estimators of the $f_{\boldsymbol{a}}$ in (20) using Theorem 1 and the assumption that $E(\epsilon_{\boldsymbol{x}}) = 0$,

$$\hat{f}_{\boldsymbol{a}} = \frac{1}{q^m} \sum_{\boldsymbol{x} \in C^{\perp}} y(\boldsymbol{x}) \mathcal{X}^*_{\boldsymbol{a}}(\boldsymbol{x}). \tag{23}$$

Then, we can easily estimate the effects using Fourier transforms. There are Fourier transforms softwares to calculate (23) for any monotonic set A.

In particular, when $q = 2^i$ where i is a integer and $i \geq 1$, we can use the vector-radix fast Fourier transform (FFT), which is a multidimensional fast Fourier transform, to calculate (23) for all $\boldsymbol{a} \in I_A$. The complexity of vector-radix FFT is $O(q^m \log q^m)$.

5 Relation Between the Fourier Coefficients and the Effects

In this section, I refer the theorems of the relation between the Fourier coefficients and the Effects (the general mean, the main factor and the interaction).

First, a theorem of the relation between the Fourier coefficient and the general mean was given as follows.

Theorem 2 *[9]*

Let $\hat{\mu}$ *be the unbiased estimator of the general mean* μ *in the model of Sect.3.1, and let* $\hat{f}_{0\ldots0}$ *be that of the Fourier coefficient* $f_{0\ldots0}$ *in the model of Sect.4.1.*

Then, the following equation holds:

$$\hat{\mu} = \hat{f}_{0\ldots0}. \tag{24}$$

□

Second, a theorem of the relation between the Fourier coefficients and the effect of the main factor was given as follows.

Theorem 3 *[9]*

Let $\hat{\alpha}_i(\varphi)$ *be the unbiased estimator of the effect of the main factor* $\alpha_i(\varphi)$ *in the model of Sect.3.1, and let* $\hat{f}_{0\ldots0a_i0\ldots0}$ *be that of the Fourier coefficient* $f_{0\ldots0a_i0\ldots0}$ *in the model of Sect.4.1.*

Then, the following equation holds:

$$\hat{\alpha}_i(\varphi) = \sum_{\substack{a_i \in GF(q) \\ a_i \neq 0}} \mathcal{X}_{a_i}(\varphi) \hat{f}_{0\ldots0a_i0\ldots0}. \tag{25}$$

□

Next, a theorem of the relation between the Fourier coefficients and the effect of the 2-factor interaction was given as follows.

Theorem 4 *[9]*

Let $\hat{\beta}_{i,j}(\varphi,\psi)$ *be the unbiased estimator of the effect of the interaction* $\beta_{i,j}(\varphi,\psi)$ *in the model of Sect.3.1, and let* $\hat{f}_{0\ldots0a_i0\ldots0a_j0\ldots0}$ *be that of the Fourier coefficient* $f_{0\ldots0a_i0\ldots0a_j0\ldots0}$ *in the model of Sect.4.1.*

Then, the following equation holds:

$$\hat{\beta}_{i,j}(\varphi,\psi) = \sum_{\substack{a_i \in GF(q) \\ a_i \neq 0}} \sum_{\substack{a_j \in GF(q) \\ a_j \neq 0}} \mathcal{X}_{a_i}(\varphi) \mathcal{X}_{a_j}(\psi) \hat{f}_{0\ldots0a_i0\ldots0a_j0\ldots0}. \tag{26}$$

□

Last, a theorem of the relation between the Fourier coefficients and the effect of the 2-factor interaction was given as follows.

Theorem 5 *[10]*

Let $\hat{\gamma}_{i,j,k}(\varphi,\psi,\xi)$ be the unbiased estimator of the effect of the interaction $\gamma_{i,j,k}(\varphi,\psi,\xi)$ in the model of Sect.3.1, and let $\hat{f}_{0\ldots0a_i0\ldots0a_j0\ldots0a_k0\ldots0}$ be that of the Fourier coefficient $f_{0\ldots0a_i0\ldots0a_j0\ldots0a_k0\ldots0}$ in the model of Sect.4.1.

Then, the following equation holds:

$$\hat{\gamma}_{i,j,k}(\varphi,\psi,\xi)=\sum_{\substack{a_i\in GF(q)\\a_i\neq0}}\sum_{\substack{a_j\in GF(q)\\a_j\neq0}}\sum_{\substack{a_k\in GF(q)\\a_k\neq0}}\mathcal{X}_{a_i}(\varphi)\mathcal{X}_{a_j}(\psi)\mathcal{X}_{a_k}(\xi)\hat{f}_{0\ldots0a_i0\ldots0a_j0\ldots0a_k0\ldots0}. \tag{27}$$

□

Hence, by these theorems, the effects as for the general mean, the main factor and the interaction can be obtained from the computed Fourier coefficients.

However, because the relation about the interaction are complicated, it is often difficult to obtain the interaction effects from the Fourier coefficients.

6 Relation Between the Fourier Coefficients and the Effects expressed by using the Kronecker product

In this section, I give the new relations between the Fourier coefficients and the effects expressed by using the Kronecker product.

Equation (25) can be written as follows.

$$\begin{bmatrix}\hat{\alpha}_i(0)\\\hat{\alpha}_i(1)\\\vdots\\\hat{\alpha}_i(q-1)\end{bmatrix}=\begin{bmatrix}\mathcal{X}_1(0)&\cdots&\mathcal{X}_{q-1}(0)\\\mathcal{X}_1(1)&\cdots&\mathcal{X}_{q-1}(1)\\\vdots&&\vdots\\\mathcal{X}_1(q-1)&\cdots&\mathcal{X}_{q-1}(q-1)\end{bmatrix}\begin{bmatrix}\hat{f}_{0\ldots010\ldots0}\\\hat{f}_{0\ldots020\ldots0}\\\vdots\\\hat{f}_{0\ldots0q-10\ldots0}\end{bmatrix}, \tag{28}$$

Equation (26) can be written as follows.

$$\begin{bmatrix}\hat{\beta}_{i,j}(0,0)\\\hat{\beta}_{i,j}(0,1)\\\vdots\\\hat{\beta}_{i,j}(q-1,q-1)\end{bmatrix}=\begin{bmatrix}\mathcal{X}_1(0)&\cdots&\mathcal{X}_{q-1}(0)\\\mathcal{X}_1(1)&\cdots&\mathcal{X}_{q-1}(1)\\\vdots&&\vdots\\\mathcal{X}_1(q-1)&\cdots&\mathcal{X}_{q-1}(q-1)\end{bmatrix}\otimes\begin{bmatrix}\mathcal{X}_1(0)&\cdots&\mathcal{X}_{q-1}(0)\\\mathcal{X}_1(1)&\cdots&\mathcal{X}_{q-1}(1)\\\vdots&&\vdots\\\mathcal{X}_1(q-1)&\cdots&\mathcal{X}_{q-1}(q-1)\end{bmatrix}\begin{bmatrix}\hat{f}_{0\ldots010\ldots010\ldots0}\\\hat{f}_{0\ldots010\ldots020\ldots0}\\\vdots\\\hat{f}_{0\ldots0q-10\ldots0q-10\ldots0}\end{bmatrix}, \tag{29}$$

Equation (27) can be written as follows.

$$\begin{bmatrix}\hat{\gamma}_{i,j,k}(0,0,0)\\\hat{\gamma}_{i,j,k}(0,0,1)\\\vdots\\\hat{\gamma}_{i,j,k}(q-1,q-1,q-1)\end{bmatrix}=\begin{bmatrix}\mathcal{X}_1(0)&\cdots&\mathcal{X}_{q-1}(0)\\\mathcal{X}_1(1)&\cdots&\mathcal{X}_{q-1}(1)\\\vdots&&\vdots\\\mathcal{X}_1(q-1)&\cdots&\mathcal{X}_{q-1}(q-1)\end{bmatrix}\otimes\begin{bmatrix}\mathcal{X}_1(0)&\cdots&\mathcal{X}_{q-1}(0)\\\mathcal{X}_1(1)&\cdots&\mathcal{X}_{q-1}(1)\\\vdots&&\vdots\\\mathcal{X}_1(q-1)&\cdots&\mathcal{X}_{q-1}(q-1)\end{bmatrix}$$

$$\otimes \begin{bmatrix} \mathcal{X}_1(0) & \cdots & \mathcal{X}_{q-1}(0) \\ \mathcal{X}_1(1) & \cdots & \mathcal{X}_{q-1}(1) \\ \vdots & & \vdots \\ \mathcal{X}_1(q-1) & \cdots & \mathcal{X}_{q-1}(q-1) \end{bmatrix}$$

$$\begin{bmatrix} \hat{f}_{0\ldots010\ldots010\ldots010\ldots0} \\ \hat{f}_{0\ldots010\ldots010\ldots020\ldots0} \\ \vdots \\ \hat{f}_{0\ldots0q-10\ldots0q-10\ldots0q-10\ldots0} \end{bmatrix}, \qquad (30)$$

Example 7 *Let $q = 3$ and $n = 5$. Consider the general mean, the effect of main factor F_1, and the effect of the interaction of F_1 and F_2.*

Then, from [4],

$$\mathcal{X}_l(k) = e^{2\pi ilk/3}. \qquad (31)$$

Define

$$A = \begin{bmatrix} 1 & 1 \\ e^{2\pi i/3} & e^{4\pi i/3} \\ e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}. \qquad (32)$$

Then, using (28) and (31), the equation

$$\begin{bmatrix} \hat{\alpha}_1(0) \\ \hat{\alpha}_1(1) \\ \hat{\alpha}_1(2) \end{bmatrix} = A \begin{bmatrix} \hat{f}_{10000} \\ \hat{f}_{20000} \end{bmatrix}, \qquad (33)$$

hold. Hence, it is clear that the effects of main factor F_1 (3 parameters) can be obtained from the computed Fourier coefficients (2 parameters).

Next, using (29) and (31), the following equation

$$\begin{bmatrix} \hat{\beta}_{1,2}(0,0) \\ \hat{\beta}_{1,2}(0,1) \\ \hat{\beta}_{1,2}(0,2) \\ \hat{\beta}_{1,2}(1,0) \\ \hat{\beta}_{1,2}(1,1) \\ \hat{\beta}_{1,2}(1,2) \\ \hat{\beta}_{1,2}(2,0) \\ \hat{\beta}_{1,2}(2,1) \\ \hat{\beta}_{1,2}(2,2) \end{bmatrix} = A \otimes A \begin{bmatrix} \hat{f}_{11000} \\ \hat{f}_{12000} \\ \hat{f}_{21000} \\ \hat{f}_{22000} \end{bmatrix}, \qquad (34)$$

hold. Hence, it is clear that the effects of the interaction of F_1 and F_2 (9 parameters) can be obtained from the computed Fourier coefficients (4 parameters).

Last, using (30) and (31), the following equation

$$\begin{bmatrix} \hat{\gamma}_{1,2,3}(0,0,0) \\ \hat{\gamma}_{1,2,3}(0,0,1) \\ \hat{\gamma}_{1,2,3}(0,0,2) \\ \hat{\gamma}_{1,2,3}(0,1,0) \\ \hat{\gamma}_{1,2,3}(0,1,1) \\ \hat{\gamma}_{1,2,3}(0,1,2) \\ \hat{\gamma}_{1,2,3}(0,2,0) \\ \hat{\gamma}_{1,2,3}(0,2,1) \\ \hat{\gamma}_{1,2,3}(0,2,2) \\ \hat{\gamma}_{1,2,3}(1,0,0) \\ \hat{\gamma}_{1,2,3}(1,0,1) \\ \hat{\gamma}_{1,2,3}(1,0,2) \\ \hat{\gamma}_{1,2,3}(1,1,0) \\ \hat{\gamma}_{1,2,3}(1,1,1) \\ \hat{\gamma}_{1,2,3}(1,1,2) \\ \hat{\gamma}_{1,2,3}(1,2,0) \\ \hat{\gamma}_{1,2,3}(1,2,1) \\ \hat{\gamma}_{1,2,3}(1,2,2) \\ \hat{\gamma}_{1,2,3}(2,0,0) \\ \hat{\gamma}_{1,2,3}(2,0,1) \\ \hat{\gamma}_{1,2,3}(2,0,2) \\ \hat{\gamma}_{1,2,3}(2,1,0) \\ \hat{\gamma}_{1,2,3}(2,1,1) \\ \hat{\gamma}_{1,2,3}(2,1,2) \\ \hat{\gamma}_{1,2,3}(2,2,0) \\ \hat{\gamma}_{1,2,3}(2,2,1) \\ \hat{\gamma}_{1,2,3}(2,2,2) \end{bmatrix} = A \otimes A \otimes A \begin{bmatrix} \hat{f}_{11100} \\ \hat{f}_{11200} \\ \hat{f}_{12100} \\ \hat{f}_{12200} \\ \hat{f}_{21100} \\ \hat{f}_{21200} \\ \hat{f}_{22100} \\ \hat{f}_{22200} \end{bmatrix}, \qquad (35)$$

hold. Hence, it is clear that the 3-factor interaction effects (27 parameters) can be obtained from the computed Fourier coefficients (8 parameters). □

Hence, using the new relations between the Fourier coefficients and the Effects by using the Kronecker product, the interaction effects can be easily obtained from the Fourier coefficients.

Moreover, it is clear that the matrix A is used in calculating the effects of all main factor and interaction. It means that we can reduce the amount of memory needed in the calculation.

7 Conclusion

In this paper, I have presented that the relation between the Fourier coefficients and the interaction effects in the experimental design can be more easily expressed by using Kronecker product. From the result, the interaction effects can be easily obtained from the Fourier coefficients. Therefore, it is possible to implement the estimation procedures easily as well as to understand how any interaction affects the response variable in the model based on an orthonormal system.

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References

[1] A.S. Hedayat, N.J.A. Sloane and J. Stufken, *Orthogonal Arrays: Theory and Applications*, Springer, 1999.

[2] T. Okuno and T. Haga, *Experimental Designs*, Baifukan, Tokyo, 1969.

[3] R.S. Stankovic and J. Astola, "Reading the Sampling Theorem in Multiple-Valued Logic: A journey from the (Shannon) sampling theorem to the Shannon decomposition rule," in *Proc. 37th Int. Symp. on Multiple-Valued Logic*, Oslo, Norway, May 2007.

[4] E.M. Stein, R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.

[5] I. Takahashi, *Combinatorial Theory and its Application*, Iwanami Syoten, Tokyo, 1979.

[6] H. Toutenburg and Shalabh, *Statistical Analysis of Designed Experiments* (Third Edition), Springer, 2009.

[7] Y. Ukita, T. Saito, T. Matsushima and S. Hirasawa, "A Note on the Relation between a Sampling Theorem for Functions over a $GF(q)^n$ Domain and Linear Codes," in *Proc. 2009 IEEE Int. Conf. on Systems, Man, and Cybernetics*, pp.2665–2670, San Antonio, USA, Oct. 2009.

[8] Y. Ukita, T. Saito, T. Matsushima and S. Hirasawa, "A Note on a Sampling Theorem for Functions over $GF(q)^n$ Domain," *IEICE Trans. Fundamentals*, Vol.E93-A, no.6, pp.1024-1031, June 2010.

[9] Y. Ukita and T. Matsushima, "A Note on Relation between the Fourier Coefficients and the Effects in the Experimental Design," in *Proc. 8th Int. Conf. on Information, Communications and Signal Processing*, Singapore, Dec, 2011.

[10] Y. Ukita, T. Matsushima and S. Hirasawa, "A Note on Relation Between the Fourier Coefficients and the Interaction Effects in the Experimental Design," in *Proc. 4th Int. Conf. on Intelligent and Advanced Systems*, Kuala Lumpur, Malaysia, pp.604-609, June 2012.

[11] Y. Ukita, T. Matsushima and S. Hirasawa, "A Study on the Degrees of Freedom in an Experimental Design Model Based on an Orthonormal System," *IEICE Trans. Fundamentals*, Vol.E96-A, no.2, pp.658-662, Feb. 2013.