An Equivalence between the Kaminski Hierarchy and the Barua Hierarchy

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Abstract

In this paper, we argue several decompositions of ω-regular sets into rational $\mathbf{G}_\delta$ sets. We measure the complexity of ω-regular sets by the number of rational $\mathbf{G}_\delta$ sets obtained by the decompositions. Barua (1992) studied a hierarchy $\mathcal{R}_n(n = 1, 2, 3, \cdots)$, where $\mathcal{R}_n$ is a class of ω-regular sets which are decomposed into n rational $\mathbf{G}_\delta$ sets forming a decreasing sequence. On the other hand, Kaminski (1985) defined a hierarchy $\mathbf{B}_m(m = 1, 2, 3, \cdots)$, where $\mathbf{B}_m$ is a class of ω-regular sets which are decomposed into $2m$ rational $\mathbf{G}_\delta$ sets not necessarily forming a decreasing sequence. As a main result, we claim that $\mathcal{R}_{2n} = \mathbf{B}_n$ in spite of the differences of defining conditions.

1 Introduction

Since the original work by Büchi[2], many creative studies on ω-languages have been accomplished (cf. [5], [7], [8], [9]). Among them, Landweber[7], and Takahashi and Yamasaki[8] clarified a close relation between classes of ω-languages defined on the Borel hierarchy of finite order and acceptance conditions of finite automata scanning on ω-sequences. Sixteen years after Landweber[7], Kaminski[4] researched four classes based on ω-languages such as (1), where A_i's and B_i's are ω-languages accepted by deterministic Büchi-automata. Hereafter, we call A_i, B_i the component sets of L.

$$L = \bigcup_{i=1}^{m} (A_i - B_i) \tag{1}$$

On the other hand, by applying the resolution theorem of ambiguous sets to $\underset{\sim}{\Delta}{}^0_3$ (cf. Kuratowski [6, §37. III]), Barua[1] constructed a class of ω-regular sets $\mathcal{R}_n$, where $\mathcal{R}_{n+1}$ $(n \geq 0)$ is a class of ω-languages L satisfying (iv) of the following proposition.

Proposition. (Barua [1, Theorem 3.3]) *Let $L \subseteq \Sigma^\omega$ be an ω-language on a finite alphabet Σ. Then the followings are equivalent:*
(i) *L is ω-regular;*
(ii) *L is a finite union of differences of rational $\mathbf{G}_\delta$ sets;*
(iii) *L is a finite disjoint union of differences of rational $\mathbf{G}_\delta$ sets;*
(iv) *There exists a decreasing sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n$ of rational $\mathbf{G}_\delta$ sets such that*

$$L = \bigcup_{i:even}^{n} (G_i - G_{i+1}) .$$

Here we focus on the differences between (ii) and (iv) of the above-mentioned proposition. In (ii) an ω-language L takes the form given in (1), which is discussed by Kaminski[4], and Thomas[9]. Note that the component sets $A_1, B_1, \cdots, A_m, B_m$ in the form (1) are not necessarily linearly ordered w.r.t. the inclusion relation, whereas the component sets $G_0, G_1, \cdots, G_n$ in (iv) of the proposition are in decreasing order. Therefore, the following problem arises. What kinds of relations are there between an index m in the form (1) and an index n in (iv) of the above proposition?

In section 2 of this paper, we prepare basic definitions and notations. In section 3, as a solution to the above-mentioned problem, we show the following theorem.

Theorem. *L is in $\mathcal{R}_{2m}$ if and only if L is expressed in the form* (1).

In other words, we claim that the decreasing property of component sets is dispensable.

2 Preliminary and background

Let Σ be an alphabet containing at least two elements. We denote the set of all words over Σ including the empty word ε by Σ^*. Σ^* without ε is denoted by Σ^+. Let ω be the set of all natural numbers. A mapping from ω to Σ is called an ω-word over Σ. By Σ^ω we denote the set of all ω-words over Σ. An ω-word $\alpha \in \Sigma^\omega$ is written as $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ where $\alpha_i = \alpha(i)$ $(i = 0, 1, 2, \cdots)$. We call a subset of Σ^* (Σ^ω, resp) a language (ω-language) over Σ. For $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^* \cup \Sigma^\omega$, we define the catenation of A and B as

$$AB = \{xy \in \Sigma^* \cup \Sigma^\omega |\ x \in A,\ y \in B\}.$$

The ω-power of $L \subseteq \Sigma^*$ is an ω-language defined as

$$L^\omega = \{x_0x_1x_2\cdots \in \Sigma^\omega \mid x_i \in L - \{\varepsilon\}\ \ for\ \ all\ \ i \in \omega\}.$$

For $x \in \Sigma^*$ and $z \in \Sigma^* \cup \Sigma^\omega$, if $z = xy$ for some $y \in \Sigma^* \cup \Sigma^\omega$, x is called an *initial segment* of z, and we denote the relation by $x < z$.

Definition 2.1. For each $x \in \Sigma^*$, we define an *open base* for x as follows:

$$N_x = \{\alpha \in \Sigma^\omega \mid x < \alpha\}.$$

An ω-language $A \subseteq \Sigma^\omega$ is an *open* set of the product topology on Σ^ω if $A = \cup_{x \in B} N_x$ for some $B \subseteq \Sigma^*$. An ω-language is *closed* if its complement is open. Let $\mathbf{G}$ ($\mathbf{F}$) denote the set of all open (closed) sets. $\mathbf{F}_\sigma$ ($\mathbf{G}_\delta$) is the set of all denumerable unions (intersections) of closed (open) sets. $\mathbf{G}_{\delta\sigma}$ ($\mathbf{F}_{\sigma\delta}$) is the set of all denumerable unions (intersections) of $\mathbf{G}_\delta$ ($\mathbf{F}_\sigma$) sets, respectively. The rest of the *Borel hierarchy* is defined in the same manner.

Definition 2.2. If an ω-language L is represented as

$$L = \bigcup_{i=1}^{n} A_i B_i^\omega$$

for a natural number $n \geq 1$ and regular sets $A_1, \cdots, A_n, B_1, \cdots, B_n$, then L is called an ω-*regular* set. We denote the set of all ω-regular sets by $\mathbf{REG}^\omega$.

Definition 2.3. For a given Σ-table $M = \langle Q, \Sigma, \delta, q_0 \rangle$, we define the following sets:

For $q \in Q$ and $u \in \Sigma^+$, let

$$R(q, u) = \{\delta(q, v) \mid v < u\}.$$

We also define

$$\mathcal{M}_q = \{R(q, u) \mid \delta(q, u) = q\ \ for\ \ some\ \ u \in \Sigma^+\},$$

and set $\mathcal{R}(M) = \bigcup_{q \in Q} \mathcal{M}_q$.

Definition 2.4. Given a Σ-table $M = \langle Q, \Sigma, \delta, q_0 \rangle$ and an ω-word $\alpha \in \Sigma^\omega$, the *run* r of M on α is a mapping from ω to Q such that

$$r(0) = q_0,$$

$$r(n+1) = \delta\left(r(n), \alpha(n)\right) \qquad \text{for } n \geq 0.$$

Then we formulate a set of states occurring infinitely many times while M runs on $\alpha \in \Sigma^\omega$, as follows:

$$In(\alpha, M) = \{q \in Q \mid \mathbf{card}(r^{-1}(q)) = \aleph_0\}.$$

Given a finite automaton $\langle M, F \rangle$, we call the ω-language

$$L(\langle M, F \rangle) = \{\alpha \in \Sigma^\omega \mid In(\alpha, M) \cap F \neq \phi\}$$

(which is Büchi-accepted by $\langle M, F \rangle$) a *rational* $\mathbf{G}_\delta$ set. A *rational* $\mathbf{F}_\sigma$ set is a set whose complement is a rational $\mathbf{G}_\delta$ set. We denote the set of all rational $\mathbf{G}_\delta$ sets ($\mathbf{F}_\sigma$ sets, resp) by $\mathcal{O}_3$ ($\mathcal{O}_4$) (cf. Kobayashi et al.[5]).

For a given Σ-table $M = \langle Q, \Sigma, \delta, q_0 \rangle$ and a family of state sets $\mathcal{F} \subseteq \mathcal{R}(M)$, we call $\langle M, \mathcal{F} \rangle$ a Muller automaton. Given a Muller automaton $\langle M, \mathcal{F} \rangle$, we define the ω-language Muller-accepted by $\langle M, \mathcal{F} \rangle$ as follows:

$$L(\langle M, \mathcal{F} \rangle) = \{\alpha \in \Sigma^\omega \mid In(\alpha, M) \in \mathcal{F}\}.$$

The following is a well-known result. For the proof, see, e.g., Eilenberg[3].

Proposition 2.5. *L is an ω-regular set if and only if $L = L(\langle M, \mathcal{F} \rangle)$ for some Muller automaton $\langle M, \mathcal{F} \rangle$.*

Kaminski[4] studied the following four classes.

Definition 2.6. We define four classes $\mathbf{RB}_n$, $\mathbf{B}_n$, $\mathbf{LB}_n$ and $\mathbf{LRB}_n$ of ω-regular sets as follows.

(a) $L \in \mathbf{RB}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $A_1, B_1, \cdots, A_{n-1}, B_{n-1}, A_n$ such that

$$L = \bigcup_{i=1}^{n-1} (A_i - B_i) \cup A_n.$$

(b) $L \in \mathbf{B}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $A_1, B_1, \cdots, A_n, B_n$ such that

$$L = \bigcup_{i=1}^{n} (A_i - B_i)\,.$$

(c) $L \in \mathbf{LB}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $B_1, A_2, B_2, \cdots, A_n, B_n$ such that

$$L = \overline{B}_1 \cup \bigcup_{i=2}^{n} (A_i - B_i)\,,$$

(d) $L \in \mathbf{LRB}_n$ $(n \geq 2)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $B_1, A_2, B_2, \cdots, A_{n-1}, B_{n-1}, A_n$ such that

$$L = \overline{B}_1 \cup \bigcup_{i=2}^{n-1} (A_i - B_i) \cup A_n.$$

Let $\mathcal{G} = \mathcal{R}(M) - \mathcal{F}$ for a given Muller automaton $\langle M, \mathcal{F} \rangle$. Then the following theorem holds, according to Kaminski[4, 2.12 Theorem, 2.11 Definition, and 3.8 Lemma].

Theorem 2.7. *For $n \geq 0$ the following hold:*

(a) $L(\langle M, \mathcal{F} \rangle) \in \mathbf{RB}_{n+1}$

$$\longleftrightarrow \neg\,[\exists F_1, \cdots, F_{n+1} \in \mathcal{F} \exists G_1, \cdots, G_{n+1} \in \mathcal{G}\,(F_1 \subset G_1 \subset \cdots \subset F_{n+1} \subset G_{n+1})]$$

(b) $L(\langle M, \mathcal{F} \rangle) \in \mathbf{B}_{n+1}$

$$\longleftrightarrow \neg\,[\exists F_1, \cdots, F_{n+2} \in \mathcal{F} \exists G_1, \cdots, G_{n+1} \in \mathcal{G}\,(F_1 \subset G_1 \subset \cdots \subset G_{n+1} \subset F_{n+2})]$$

(c) $L(\langle M, \mathcal{F}\rangle) \in \mathbf{LB}_{n+1}$

$\longleftrightarrow \neg\, [\exists F_1, \cdots, F_{n+1} \in \mathcal{F} \exists G_1, \cdots, G_{n+1} \in \mathcal{G}\, (G_1 \subset F_1 \subset \cdots \subset G_{n+1} \subset F_{n+1})]$

(d) $L(\langle M, \mathcal{F}\rangle) \in \mathbf{LRB}_{n+2}$

$\longleftrightarrow \neg\, [\exists F_1, \cdots, F_{n+1} \in \mathcal{F} \exists G_1, \cdots, G_{n+2} \in \mathcal{G}\, (G_1 \subset F_1 \subset \cdots \subset F_{n+1} \subset G_{n+2})]$

where $\subset$ denotes the strict inclusion.

Furthermore, the following hierarchy theorem has been obtained by Kaminski[4].

Theorem 2.8. (Kaminski[4, 4.1 Theorem])

$$\begin{tikzcd}
\mathbf{RB}_1 \arrow[r] & \mathbf{B}_1 \arrow[r] \arrow[dr] & \mathbf{RB}_2 \arrow[r] & \mathbf{B}_2 \arrow[r] \arrow[dr] & \mathbf{RB}_3 \arrow[r] & \mathbf{B}_3 \arrow[r] \arrow[dr] & \cdots \\
\mathbf{LB}_1 \arrow[r] & \mathbf{LRB}_2 \arrow[r] \arrow[ur] & \mathbf{LB}_2 \arrow[r] & \mathbf{LRB}_3 \arrow[r] \arrow[ur] & \mathbf{LB}_3 \arrow[r] & \mathbf{LRB}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

($\longrightarrow$ *expresses the strict inclusion.*)

On the basis of the Büchi-McNaughton theorem, we can conclude that $\mathbf{REG}^\omega \subseteq \mathbf{F}_{\sigma\delta} \cap \mathbf{G}_{\delta\sigma}$. Accordingly, by restricting the number of quantifiers to 2 in the theorem of Kuratowski[6, §37. III], we obtain the following corollary.

Corollary 2.9. *A set $A \subseteq \Sigma^\omega$ is in both $\mathbf{F}_{\sigma\delta}$ and $\mathbf{G}_{\delta\sigma}$ if and only if there exists a countable transfinite ordinal μ such that*

$$A = \bigcup_{\lambda:even}^{\mu} (G_\lambda - G_{\lambda+1})$$

with decreasing sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_\mu$, where each G_λ is a $\mathbf{G}_\delta$ set in Σ^ω. Here if μ is even, let $G_{\mu+1} = \phi$.

Corresponding to the ordinal number μ, Barua[1] defined the class $\mathcal{D}_{\mu+1}$ which consists of such set A's as mentioned in Corollary 2.9. In particular, $\mathcal{D}_1 = \mathbf{G}_\delta$. He constructed a class $\mathcal{R}_n (n \geq 1)$ of ω-regular sets taking the finite ordinal $n \in \omega$ as μ, as follows.

Definition 2.10. For each $n \geq 0$, L is in $\mathcal{R}_{n+1}$ iff there is a decreasing sequence of rational $\mathbf{G}_\delta$ sets $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n$ such that

$$L = \bigcup_{i:even}^{n} (G_i - G_{i+1})\,.$$

$G_0, G_1, \cdots, G_n$ are called the *component* sets of L. In particular, $\mathcal{R}_1 = \mathcal{O}_3$.

Proposition 2.11. (Barua[1, Theorem 4.7])
For $n \geq 1$ $\quad \mathcal{R}_n = \mathcal{D}_n \cap \mathbf{REG}^\omega$.

This proposition is an extension of Landweber's theorem; $\mathcal{O}_3 = \mathbf{G}_\delta \cap \mathbf{REG}^\omega$ (cf. Landweber[7]).

3 The Kaminski hierarchy and the Barua hierarchy

In this section, for each $m \geq 1$ we newly define a class $\mathcal{L}_m$ which is the dual class of the Barua class $\mathcal{R}_m$. We then show that the Barua class $\mathcal{R}_m$ is identical with $\mathbf{RB}_n$ or $\mathbf{B}_n$ for some $n \geq 1$, according as m is odd or even. Similar results can be obtained with $\mathcal{L}_m$, $\mathbf{LB}_n$, and $\mathbf{LRB}_{n+1}$ instead of $\mathcal{R}_m$, $\mathbf{RB}_n$, and $\mathbf{B}_n$.

Definition 3.1. Consider a Muller automaton $\langle M, \mathcal{H}\rangle$, where $M = \langle Q, \Sigma, \delta, q_0\rangle$ and $\mathcal{H} \subseteq \mathcal{R}(M)$. For $\mathcal{H} \subseteq \mathcal{R}(M)$, we define $\widehat{\mathcal{H}}$ (called the *cyclic closure* of $\mathcal{H}$) and $\widetilde{\mathcal{H}}$ as follows:

$$\widehat{\mathcal{H}} = \{H_1 \cup H_2 \mid H_1 \in \mathcal{H} \cap \mathcal{M}_q \text{ and } H_2 \in \mathcal{M}_q \text{ for some } q \in Q\}.$$

$$\widetilde{\mathcal{H}} = \{H_1 \cup H_2 \mid H_2 \not\subseteq H_1 \text{ and } H_1 \in \mathcal{H} \cap \mathcal{M}_q \text{ and } H_2 \in \mathcal{M}_q \text{ for some } q \in Q\}.$$

Note that $\widehat{\mathcal{H}} = \widetilde{\mathcal{H}} \cup \mathcal{H}$, and recall that $\mathcal{M}_q$ is a set of nonempty subsets of Q. Then the following holds.

Lemma 3.2. (Landweber[7])

$$\widehat{\mathcal{F}} \subseteq \mathcal{F} \qquad \textit{iff} \qquad L(\langle M, \mathcal{F}\rangle) \in \mathcal{R}_1.$$

Definition 3.3. Fix a natural number $n \geq 0$. For a Muller automaton $\langle M, \mathcal{F}\rangle$, let $\mathcal{G} = \mathcal{R}(M) - \mathcal{F}$. Then we inductively define cyclic closures $\mathcal{F}_i$, $\mathcal{G}_i$ $\quad (i = 0, 1, \cdots, n)$ as follows.

Basis. $\mathcal{F}_0 = \widehat{\mathcal{F}}$, $\mathcal{G}_0 = \widehat{\mathcal{G}}$

Inductive step. For $i = 0, 1, \cdots, n-1$, set

$$\mathcal{F}_{i+1} = \mathcal{F}_i \widehat{\cap} \mathcal{H}_i \ , \qquad \mathcal{G}_{i+1} = \mathcal{G}_i \widehat{\cap \overline{\mathcal{H}}_i} \ ,$$

$$\text{where} \quad \mathcal{H}_i = \begin{cases} \mathcal{F} & \text{if } i \text{ is odd} \\ \mathcal{G} & \text{if } i \text{ is even.} \end{cases}$$

Concerning $\mathcal{F}_i$, Barua[1] has already obtained the following result.

Proposition 3.4. (Barua[1, Theorem 5.1])
For $n \geq 0$, the following hold:

(a) $L(\langle M, \mathcal{F} \rangle) \in \mathcal{R}_{2n+1}$ *iff* $\mathcal{F}_{2n} \cap \mathcal{G} = \phi$.

(b) $L(\langle M, \mathcal{F} \rangle) \in \mathcal{R}_{2n+2}$ *iff* $\mathcal{F}_{2n+1} \cap \mathcal{F} = \phi$.

Since Proposition 3.4 (a) for $n = 0$ is the same as Lemma 3.2, Proposition 3.4 is a generalization of the lemma. In this way, Barua[1] obtained two points (a) and (b) for the class $\mathcal{R}_n$. On the other hand, Kaminski[4] obtained four points (a), (b), (c), and (d), as seen in Theorem 2.7. Thus we define a class $\mathcal{L}_n$ as the dual class of $\mathcal{R}_n$, namely

Definition 3.5. For each $n \geq 0$ define a class of ω-regular sets $\mathcal{L}_{n+1}$ as follows.

L is in $\mathcal{L}_{n+1}$ *iff* there exist rational $\mathbf{G}_\delta$ sets $G_0, G_1, \cdots, G_n$such that

$$G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \qquad \text{and}$$

$$L = \overline{G}_0 \cup \bigcup_{i:odd}^{n} (G_i - G_{i+1}) \,.$$

Lemma 3.6.

$L \in \mathcal{L}_n$ *iff* $\overline{L} \in \mathcal{R}_n$.

Proof. The following equivalence holds if L is in $\mathcal{L}_n$.

$$L = (\Sigma^\omega - H_0) \cup \bigcup_{i:odd}^{n-1} (H_i - H_{i+1}) \quad \textit{iff} \quad \overline{L} = \bigcup_{i:even}^{n-1} (H_i - H_{i+1}) \,.$$

∎

Theorem 3.7. *For $n \geq 0$ the following hold.*

(a) $L(\langle M, \mathcal{F}\rangle) \in \mathcal{R}_{2n+1}$ *iff* $\mathcal{F}_{2n} \cap \mathcal{G} = \phi$.

(b) $L(\langle M, \mathcal{F}\rangle) \in \mathcal{R}_{2n+2}$ *iff* $\mathcal{F}_{2n+1} \cap \mathcal{F} = \phi$.

(c) $L(\langle M, \mathcal{F}\rangle) \in \mathcal{L}_{2n+1}$ *iff* $\mathcal{G}_{2n} \cap \mathcal{F} = \phi$.

(d) $L(\langle M, \mathcal{F}\rangle) \in \mathcal{L}_{2n+2}$ *iff* $\mathcal{G}_{2n+1} \cap \mathcal{G} = \phi$.

Proof. Points (a) and (b) are due to Barua[1] and already presented as Proposition 3.4. We here rewrite (a) and (b), because later we want to discuss the four points (a), (b), (c), and (d) as in Theorem 2.7.

Since points (a) and (b) hold, points (c) and (d) are proved as follows using Lemma 3.6.

The proof of point (c): From point (a)

$$\mathcal{F}_{2n} \cap \mathcal{G} = \phi \qquad \textit{iff} \qquad L(\langle M, \mathcal{F}\rangle) \in \mathcal{R}_{2n+1}.$$

By Lemma 3.6

$$\mathcal{F}_{2n} \cap \mathcal{G} = \phi \qquad \textit{iff} \qquad L(\langle M, \mathcal{G}\rangle) \in \mathcal{L}_{2n+1}.$$

Replacing $\mathcal{G}$ ($\mathcal{F}_{2n}$) by $\mathcal{F}$ ($\mathcal{G}_{2n}$), we obtain

$$\mathcal{G}_{2n} \cap \mathcal{F} = \phi \qquad \textit{iff} \qquad L(\langle M, \mathcal{F}\rangle) \in \mathcal{L}_{2n+1}.$$

The proof of point (d): From point (b) and Lemma 3.6, we obtain point (d) by the same argument as for point (c). ∎

Our main result is Theorem 3.11. In order to show Theorem 3.11, we first state Lemma 3.8 and then prove Theorem 3.9 which implies Corollary 3.10. By definitions we immediately obtain the following

Lemma 3.8. *For $n \geq 0$*

(a) $\mathcal{R}_{2n+1} \subseteq \mathbf{RB}_{n+1}$,

(b) $\mathcal{R}_{2n+2} \subseteq \mathbf{B}_{n+1}$,

(c) $\mathcal{L}_{2n+1} \subseteq \mathbf{LB}_{n+1}$,

(d) $\mathcal{L}_{2n+2} \subseteq \mathbf{LRB}_{n+2}$.

Theorem 3.9. *For $n \geq 0$ there hold*

(a) $\forall G \in \mathcal{G}\ [\ G \in \mathcal{F}_{2n} \longrightarrow$
$\exists F_1, \cdots, F_{n+1} \in \mathcal{F}\ \exists G_1, \cdots, G_n \in \mathcal{G}\ (F_1 \subset G_1 \subset \cdots \subset F_{n+1} \subset G)\ ]$,

(b) $\forall F \in \mathcal{F}\ [\ F \in \mathcal{F}_{2n+1} \longrightarrow$

$$\exists F_1, \cdots, F_{n+1} \in \mathcal{F}\ \ \exists G_1, \cdots, G_{n+1} \in \mathcal{G}$$
$$(F_1 \subset G_1 \subset \cdots \subset F_{n+1} \subset G_{n+1} \subset F)\ ],$$

(c) $\forall F \in \mathcal{F}\ [\ F \in \mathcal{G}_{2n} \longrightarrow$

$$\exists F_1, \cdots, F_n \in \mathcal{F}\ \ \exists G_1, \cdots, G_{n+1} \in \mathcal{G}$$
$$(G_1 \subset F_1 \subset \cdots \subset G_{n+1} \subset F)\ ],$$

(d) $\forall G \in \mathcal{G}\ [\ G \in \mathcal{G}_{2n+1} \longrightarrow$

$$\exists F_1, \cdots, F_{n+1} \in \mathcal{F}\ \ \exists G_1, \cdots, G_{n+1} \in \mathcal{G}$$
$$(G_1 \subset F_1 \subset \cdots \subset G_{n+1} \subset F_{n+1} \subset G)\ ].$$

Proof. Point (c)[(d)] is the dual version of point (a)[(b)]. Since point (a) is proved in the same manner as point (b), we show point (b) by induction on $n \geq 0$.

Basis. Fix $F \in \mathcal{F}$ arbitrarily, and assume $F \in \mathcal{F}_1$. Since $\mathcal{F}_1 = \mathcal{F}_0 \widetilde{\cap} \mathcal{G} = (\mathcal{F}_0 \widetilde{\cap} \mathcal{G}) \cup (\mathcal{F}_0 \cap \mathcal{G})$, $F \in \mathcal{F}_0 \widetilde{\cap} \mathcal{G}$ or $F \in \mathcal{F}_0 \cap \mathcal{G}$. If $F \in \mathcal{F}_0 \cap \mathcal{G}$, then $F \in \mathcal{G}$, i.e., $F \notin \mathcal{F}$. But this is a contradiction. Therefore $F \in \mathcal{F}_0 \widetilde{\cap} \mathcal{G}$. Accordingly, we obtain the implication

$$F \in \mathcal{F}_0 \widetilde{\cap} \mathcal{G} \longrightarrow \exists F_1 \in \mathcal{F}\ \ \exists G_1 \in \mathcal{G}\ (F_1 \subset G_1 \subset F).$$

(The details are the same as the following inductive step.)

Inductive step. Let point (b) hold for an arbitrary $n \geq 0$. Then fix $F \in \mathcal{F}$ arbitrarily, and assume $F \in \mathcal{F}_{2(n+1)+1}$. Since $\mathcal{F}_{2(n+1)+1} = \widetilde{\mathcal{F}_{2(n+1)}} \cap \mathcal{G} = (\widetilde{\mathcal{F}_{2(n+1)}} \cap \mathcal{G}) \cup (\mathcal{F}_{2(n+1)} \cap \mathcal{G})$, $F \in \widetilde{\mathcal{F}_{2(n+1)}} \cap \mathcal{G}$ or $F \in \mathcal{F}_{2(n+1)} \cap \mathcal{G}$. If $F \in \mathcal{F}_{2(n+1)} \cap \mathcal{G}$, then $F \in \mathcal{G}$, i.e., $F \notin \mathcal{F}$. But this is a contradiction. Therefore $F \in \widetilde{\mathcal{F}_{2(n+1)}} \cap \mathcal{G}$. Accordingly, we obtain the following implications.

$$F \in \widetilde{\mathcal{F}_{2(n+1)}} \cap \mathcal{G}$$
$$\longrightarrow \exists p \in Q\ \exists G_{n+2}\ \exists Y [F = G_{n+2} \cup Y\ \wedge\ Y \not\subseteq G_{n+2}\ \wedge\ G_{n+2} \in (\mathcal{F}_{2n+2} \cap \mathcal{G}) \cap \mathcal{M}_p\ \wedge\ Y \in \mathcal{M}_p]$$
$$\longrightarrow \exists G_{n+2} \in \mathcal{G}\ \ (G_{n+2} \subset F\ \ \wedge\ \ G_{n+2} \in \mathcal{F}_{2n+2})$$
$$\longrightarrow \exists G_{n+2} \in \mathcal{G}\ \ (G_{n+2} \subset F\ \ \wedge\ \ G_{n+2} \in \widetilde{\mathcal{F}_{2n+1}} \cap \mathcal{F})$$
$$\longrightarrow \exists G_{n+2} \in \mathcal{G}\ \exists p \in Q\ \exists F_{n+2}\ \exists X\ [G_{n+2} \subset F\ \wedge\ G_{n+2} = F_{n+2} \cup X\ \wedge\ X \not\subseteq F_{n+2}\ \wedge\ F_{n+2} \in (\mathcal{F}_{2n+1} \cap \mathcal{F}) \cap \mathcal{M}_p\ \wedge\ X \in \mathcal{M}_p]$$
$$\longrightarrow \exists F_{n+2} \in \mathcal{F}\ \ \exists G_{n+2} \in \mathcal{G}\ \ (F_{n+2} \in \mathcal{F}_{2n+1}\ \ \wedge\ \ F_{n+2} \subset G_{n+2} \subset F)$$

$$\xrightarrow{\dagger} \exists F_1, \cdots, F_{n+1}, F_{n+2} \in \mathcal{F} \;\; \exists G_1, \cdots, G_{n+1}, G_{n+2} \in \mathcal{G}$$
$$(F_1 \subset G_1 \subset \cdots \subset F_{n+1} \subset G_{n+1} \subset F_{n+2} \subset G_{n+2} \subset F).$$

(The implication $\xrightarrow{\dagger}$ follows from the inductive hypothesis.) This completes the induction. Therefore point (b) holds. ∎

We can infer Corollary 3.10 from Theorem 3.9, because (2) below is logically true:

$$\begin{array}{ll} \textit{If} & \forall H \in \mathcal{H} \; [\; P(H) \longrightarrow Q(H) \;], \\ \textit{then} & \exists H_1 \in \mathcal{H} \; P(H_1) \longrightarrow \exists H_2 \in \mathcal{H} \; Q(H_2). \end{array} \qquad (2)$$

Corollary 3.10. *For $n \geq 0$ there hold*

(a)

$$\mathcal{F}_{2n} \cap \mathcal{G} \neq \phi$$
$$\longrightarrow$$
$$\exists F_1, \cdots, F_{n+1} \in \mathcal{F} \;\; \exists G_1, \cdots, G_{n+1} \in \mathcal{G}$$
$$(F_1 \subset G_1 \subset \cdots \subset F_{n+1} \subset G_{n+1}),$$

(b)

$$\mathcal{F}_{2n+1} \cap \mathcal{F} \neq \phi$$
$$\longrightarrow$$
$$\exists F_1, \cdots, F_{n+1}, F_{n+2} \in \mathcal{F} \;\; \exists G_1, \cdots, G_{n+1} \in \mathcal{G}$$
$$(F_1 \subset G_1 \subset \cdots \subset F_{n+1} \subset G_{n+1} \subset F_{n+2}),$$

(c)

$$\mathcal{G}_{2n} \cap \mathcal{F} \neq \phi$$
$$\longrightarrow$$
$$\exists F_1, \cdots, F_{n+1} \in \mathcal{F} \;\; \exists G_1, \cdots, G_{n+1} \in \mathcal{G}$$
$$(G_1 \subset F_1 \subset \cdots \subset G_{n+1} \subset F_{n+1}),$$

(d)

$$\mathcal{G}_{2n+1} \cap \mathcal{G} \neq \phi$$
$$\longrightarrow$$
$$\exists F_1, \cdots, F_{n+1} \in \mathcal{F} \;\; \exists G_1, \cdots, G_{n+1}, G_{n+2} \in \mathcal{G}$$
$$(G_1 \subset F_1 \subset \cdots \subset G_{n+1} \subset F_{n+1} \subset G_{n+2}).$$

We derive the main results by means of Theorem 2.7, Corollary 3.10, Theorem 3.7, and Lemma 3.8.

Theorem 3.11. *For* $n \geq 0$

(a) $\mathbf{RB}_{n+1} = \mathcal{R}_{2n+1}$,
(b) $\mathbf{B}_{n+1} = \mathcal{R}_{2n+2}$,
(c) $\mathbf{LB}_{n+1} = \mathcal{L}_{2n+1}$,
(d) $\mathbf{LRB}_{n+2} = \mathcal{L}_{2n+2}$.

Proof. We give a proof of point (b), since the proofs of points (a), (c), and (d) are similar.

Let us consider a Muller automaton $\langle M, \mathcal{F}\rangle$. We then derive the following implications from Theorem 2.7, Corollary 3.10, and Theorem 3.7.

$$L(\langle M, \mathcal{F}\rangle) \in \mathbf{B}_{n+1}$$

$$\xleftrightarrow{(1)} \neg \begin{bmatrix} \exists F_1, \cdots, F_{n+2} \in \mathcal{F} \;\; \exists G_1, \cdots, G_{n+1} \in \mathcal{G} \\ (F_1 \subset G_1 \subset \cdots \subset G_{n+1} \subset F_{n+2}) \end{bmatrix}$$

$$\xrightarrow{(2)} \quad \mathcal{F}_{2n+1} \cap \mathcal{F} = \phi$$

$$\xleftrightarrow{(3)} \quad L(\langle M, \mathcal{F}\rangle) \in \mathcal{R}_{2n+2}.$$

(1) by Theorem 2.7
(2) by Corollary 3.10
(3) by Theorem 3.7

Thus $\mathbf{B}_{n+1} \subseteq \mathcal{R}_{2n+2}$. The reverse inclusion is due to Lemma 3.8. ∎

4 Conclusion

By Theorem 3.11 we conclude that the requisite decreasing condition for the component sets of ω-languages in $\mathcal{R}_n, \mathcal{L}_n$ is not an essential property for constructing these ω-languages. However, the decrement of component sets reported by Kuratowski[6] is a very convenient property, as seen in the proof of Lemma 3.6. In other words, when we consider the four classes $\mathbf{RB}_n, \mathbf{B}_n, \mathbf{LB}_n$, and $\mathbf{LRB}_{n+1}$ proposed by Kaminski[4], it is possible for the component sets to possess the decreasing property, if occasion demands.

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